Yang-Lee zeros of the *Q*-state Potts model on recursive lattices

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The Yang-Lee zeros of the Q-state Potts model on recursive lattices are studied for noninteger values of Q. Considering one-dimensional (1D) lattice as a Bethe lattice with coordination number equal to 2, the location of Yang-Lee zeros of 1D ferromagnetic and antiferromagnetic Potts models is completely analyzed in terms of neutral periodical points. Three different regimes for Yang-Lee zeros are found for $Q>1$ and $0<Q<1$. An exact analytical formula for the equation of phase transition points is derived for the 1D case. It is shown that Yang-Lee zeros of the Q-state Potts model on a Bethe lattice are located on arcs of circles with the radius depending on Q and temperature for $Q>1$. Complex magnetic field metastability regions are studied for the $Q>1$ and $0<Q<1$ cases. The Yang-Lee edge singularity exponents are calculated for both 1D and Bethe lattice Potts models. The dynamics of metastability regions for different values of Q is studied numerically.

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I. INTRODUCTION

The Q-state Potts model plays an important role in the general theory of phase transitions and critical phenomena [1]. It was initially defined for an integer Q as a generalization of the Ising model ($Q=2$) to more-than-two components [2]. Later on, it was shown that the Potts model for noninteger values of Q may describe the properties of a number of physical systems such as dilute spin glasses [3], gelatation and vulcanization of branched polymers ($0<Q<1$) [4]. Also it was shown that the bond and site percolation problems could be formulated in terms of Potts models with pair and multisite interactions in the $Q=1$ limit.

In 1952, for the first time, Lee and Yang [5] in their famous papers studied the distribution of zeros of the partition function considered as a function of a complex magnetic filed ($e^{-2H/kT}$ activity, H is a magnetic field). They proved the circle theorem which states that zeros of the partition function of an Ising ferromagnet lie on a unit circle in the complex activity plane (*Yang-Lee zeros*). After these pioneering works of Lee and Yang, Fisher [6], in 1964, initiated the study of partition function zeros in the complex temperature plane (*Fisher zeros*). These methods were then extended to other types of interactions and found wide applications [7].

Recently, much attention was drawn to the problem of the Yang-Lee and Fisher zeros of the Q-state Potts model for both integer and noninteger values of Q. The microcanonical transfer matrix method was used to study the Yang-Lee and Fisher zeros of the noninteger Q-state Potts model in two and three dimensions [8–11].

Derrida, De Seze, and Itzykson [12] showed for the first time that the Fisher zeros in hierarchical lattice models are just the Julia set corresponding to the renormalization transformation. They found a fractal structure for the Fisher zeros in Q-state Potts model on the diamond lattice. Recently, Monroe investigated Julia sets of the Potts model on recursive lattices [13]. He found that the box counting fractal dimension of the Julia set of the governing recurrence relation is a minimum at a phase transition. This gives an alternative criterion for studying phase transitions in models defined on recursive lattices.

In 1994, Glumac and Uzelac [14], using the transfer matrix method, analytically studied the distribution of Yang-Lee zeros for the one-dimensional ferromagnetic Potts model with arbitrary and continuous $Q\geqslant 0$. For $0<Q<1$ they obtained that for high temperatures the Yang-Lee zeros lie on a real interval, and for low temperatures these are located partially on the real axis and in complex conjugate pairs on the activity plane. Later on, Monroe investigated this model by means of the dynamical systems approach and confirmed that for $Q<1$ there is a real interval of Yang-Lee zeros [15]. Then, Kim and Creswick found that for $Q>1$ the Yang-Lee zeros lie on a circle with radius R, where $R<1$ for $1<Q<2$, $R>1$ for $Q>2$, and $R=1$ for $Q=2$ [9]. However, it is not clear yet what is the location of Yang-Lee zeros for $0<Q<1$ at low temperatures [14].

In this paper the Yang-Lee zeros of the one-dimensional and Bethe lattice Q-state Potts models are studied using the dynamical systems approach. It is shown that for the one-dimensional Potts model the partition function becomes zero when the corresponding recurrence relation has neutral fixed points for a given value of magnetic field. Using this correspondence between zeros of the partition function and neutral fixed points of the recurrence relation, the Yang-Lee zeros of both ferromagnetic and antiferromagnetic Potts models are completely studied analytically. The location of Yang-Lee zeros of the ferromagnetic Potts model for $0<Q<1$ is found. Also, formulas for the density of the Yang-Lee zeros are derived and edge singularity exponents are calculated. For the Potts model on a Bethe lattice it is shown that the Yang-Lee zeros are located on a phase coexistence line in the complex magnetic field plane. Here, the phase coexistence line is defined as a line in the complex magnetic field plane, where the absolute values of the derivatives of the recurrence relation in two attracting fixed points are equal. It is worth noting that, Monroe [16] used a similar criterion for studying critical properties of the Potts model on recursive lattices. For the Bethe lattice case an analytical study of

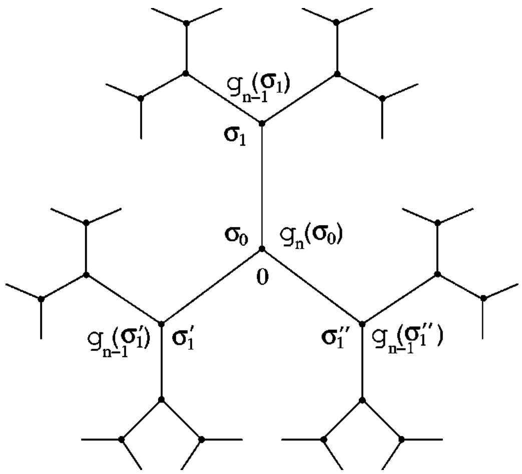

FIG. 1. The Bethe lattice with coordination number $\gamma=3$.

Yang-Lee zeros is performed also. A formula for the Yang-Lee edge singularity points is derived and edge singularity exponents are calculated. Our analytical treatment confirmed the results of numerical calculations. Further, metastability regions in a complex magnetic field plane are investigated. It is shown that the border of a metastability region may be found from the condition of existence of a neutral fixed point of the governing recurrence relation.

The structure of this paper is as follows: in Sec. II an exact recurrence relation (Potts-Bethe mapping) for the Q-state Potts model on a Bethe lattice is derived. Applying the theory of dynamical systems to the problem of phase transitions, it is shown that critical points may be associated with neutral periodical points of the corresponding mapping. In Sec. III the Yang-Lee zeros and edge singularities of ferromagnetic and antiferromagnetic Potts models are studied analytically for noninteger Q, considering a 1D lattice as a Bethe lattice with coordination number $\gamma=2$. In Sec. IV the Yang-Lee zeros and edge singularities of the Potts model on a Bethe lattice with coordination number $\gamma>2$ are studied numerically. In the final section the dynamics of complex magnetic metastability regions is studied numerically and the explanation of results is given.

II. THE Q-STATE POTTS MODEL ON THE BETHE LATTICE

The Q-state Potts model in the magnetic field is defined by the Hamiltonian

$$-\beta\mathcal{H}=J\sum_{\langle i,j\rangle}\delta(\sigma_i,\sigma_j)+h\sum_i\delta(\sigma_i,0), \tag{1}$$

where σ_i takes the values $0,1,2,\ldots,Q-1$, and $\beta=1/kT$. The first sum on the right-hand side of Eq. (1) goes over all edges and the second one over all sites on the lattice. The partition function of the model is given by

$$\mathcal{Z}=\sum_{\{\sigma\}}e^{-\beta\mathcal{H}},$$

where the summation goes over all configurations of the system.

Using the recursive structure of the Bethe lattice (Fig. 1) one can derive an exact recurrence relation and apply the theory of dynamical systems to investigate the thermodynamical properties of models defined on it [17]. Cutting the lattice at the central site 0 one will obtain γ interacting nth generation branches [18]. By denoting the partition function of the nth generation branch with the basic site 0 in the state σ_0 as $g_n(\sigma_0)$, the partition function may be written as follows:

$$\mathcal{Z}_n=\sum_{\{\sigma_0\}}\exp\{h\,\delta(\sigma_0,0)\}[g_n(\sigma_0)]^\gamma, \tag{2}$$

where σ_0 is the Potts variable at the central site 0 of the lattice (Fig. 1). Applying the "cutting" procedure to an nth generation branch one can derive the recurrence relation for $g_n(\sigma_0)$,

$$g_n(\sigma_0)=\sum_{\{\sigma_1\}}\exp\{J\,\delta(\sigma_0,\sigma_1)+h\,\delta(\sigma_1,0)\}[g_{n-1}(\sigma_1)]^{\gamma-1}. \tag{3}$$

Introducing the notation

$$x_n=\frac{g_n(\sigma\neq 0)}{g_n(\sigma=0)}, \tag{4}$$

one can obtain the Potts-Bethe map from Eq. (3),

$$x_n=f(x_{n-1}),\qquad f(x)=\frac{\mu+(z+Q-2)x^{\gamma-1}}{z\mu+(Q-1)x^{\gamma-1}}, \tag{5}$$

where $\mu=e^h$ and $z=e^J$.

The magnetization of the central site for a Bethe lattice of n generations may be written as

$$M_n=\mathcal{Z}_n^{-1}\sum_{\{\sigma_0\}}\delta(\sigma_0,0)e^{-\beta\mathcal{H}}=\frac{\mu}{\mu+(Q-1)x_n^\gamma}. \tag{6}$$

Instead of considering $M(\mu)$ in Eq. (6) it is more convenient to consider the function $\overline{M}(\mu)=2M(\mu)-1$, which has the same analytical properties as $M(\mu)$ and gives correct magnetization for the Ising model ($Q=2$)

$$\overline{M}(\mu)=\frac{\mu-(Q-1)x^\gamma}{\mu+(Q-1)x^\gamma}. \tag{7}$$

In the following, the formulas in Eqs. (5) and (6) will be generalized to noninteger values of $Q\geqslant 0$.

Let us now give some definitions and briefly discuss the problem of phase transitions on recursive lattices in terms of dynamical systems theory. The point x^* is called a periodical point with period k of the mapping $x_n=f(x_{n-1})$ if it is a solution to the equation $f^k(x)=x$. Here f^n means a superposition $f^n\equiv f\circ f\circ\cdots\circ f$. If $k=1$, x^* is called a fixed point. To analyze the stability of a periodic point x^* of period k, i.e., either iterations of $f(x)$ tend to the periodical point or move away from it, one should compute the derivative of f^k, $\lambda = (f^k)'(x)$ $(' = d/dx)$, at this point. A periodic point x^* is (1) attracting (stable) if $|\lambda| < 1$, (2) repelling (unstable) if $|\lambda| > 1$, and (3) neutral (indifferent) if $|\lambda| = 1$.

The thermodynamic properties of models defined on recursive lattices may be investigated by studying the dynamics of the corresponding recursive function. For the ferromagnetic Ising model, for example, this may be done as follows. For high temperatures $(T > T_c)$ the recursive function Eq. (5) has only one attracting fixed point corresponding to a stable paramagnetic state. For low temperatures $(T < T_c)$ the recursive function Eq. (5) has two attracting fixed points. In the absence of a magnetic field, these two attracting fixed points correspond to two possible ferromagnetic states with opposite magnetizations. It is well known that for $h = 0$ and $T < T_c$ the system undergoes a first-order phase transition and the condition $|\lambda_1| = |\lambda_2|$ is satisfied, where $|\lambda_{1,2}|$ are derivatives of $f(x)$ in these attracting fixed points. In the presence of a magnetic field, the stable state corresponds to the fixed point with maximum $|\lambda|$ and the magnetization in this state has the same direction as the external magnetic field. The other fixed point corresponds to a metastable state, which may be achieved by a sudden reversal in the sign of the magnetic field. The boundary of the metastability region may be found from the condition that one of the fixed points becomes neutral. For more details about the dynamics of metastable states see the book by Chaikin and Lubensky [19]. The critical temperature corresponds to the values of the magnetic field and temperature when the fixed points of recursive function $f(x)$ are neutral and repelling. The values of external parameters (temperature, magnetic field, etc.) at which the recursive function $f(x)$ has a neutral periodical point of period k may be obtained from the following system of equations:

$$f^k(x) = x,$$
$$|f^{k'}(x)| = 1. \tag{8}$$

Thus, for the models defined on recursive lattices, critical points and the boundary of metastability regions may be associated with neutral periodical points of the model's mapping. In the following section the Yang-Lee zeros of the 1D Q-state Potts model are investigated in terms of neutral fixed points.

III. THE YANG-LEE ZEROS OF THE 1D Q-STATE POTTS MODEL

A one-dimensional lattice may be considered as a particular case of the Bethe lattice with coordination number $\gamma = 2$. In this case the Bethe-Potts mapping (5) becomes a Möbius transformation, i.e., a rational map of the form

$$R(x) = \frac{ax+b}{cx+d}, \quad ad - bc \neq 0,$$

where

$$R(\infty) = a/c, \quad R(-d/c) = \infty,$$

if $c \neq 0$, while $R(\infty) = \infty$ when $c = 0$. The dynamics of such maps is rather simple [20]: If $f(x)$ is a Möbius transformation with two fixed points, then either $f^n(x)$ converge to one of the fixed points of $f(x)$ (one of the fixed points is attracting and the other is repelling), or they move cyclically through a finite set of points, or they form a dense subset of some circle (both fixed points are neutral). It follows from the dynamics of the Möbius transformation that phase transitions correspond to values of the temperature and magnetic field where the recursive function $f(x)$ has only neutral fixed points. Thus, to study the Yang-Lee zeros one should study the system (8) to find the values of μ at which this system has solutions for $k = 1$. For neutral fixed points the system (8) may be written in the form

$$f(x) = x,$$
$$f'(x) = e^{i\phi}, \quad \phi \in [0, 2\pi]. \tag{9}$$

Excluding x from the equations of the system (9) after some algebra one can find the equation of phase transitions in the form

$$z^2\mu^2 - 2[(z-1)(z+Q-1)\cos\phi + 1 - Q]\mu + (z+Q-2)^2 = 0, \tag{10}$$

where $\phi \in [0, 2\pi]$. Since $\cos\phi$ is an even function of ϕ, we may restrict ourselves to $\phi \in [0, \pi]$. Thus, for given z and Q, Eq. (10) is a parametric equation of the Yang-Lee zeros, where ϕ is a parameter. Analyzing Eq. (10) one can find the location of the Yang-Lee zeros of the Potts model. It is worth noting that using this equation one can study the Fisher and Potts zeros[1] in the same way.[2]

First of all, note that Eq. (10) is a quadratic equation of μ with real coefficients. Hence, solutions to this equation lie either on the real axes or in complex conjugate pairs on a circle with radius $R = |\mu| = |z + Q - 2|/z$ for any $\phi \in [0, \pi]$ (see also Ref. [9]). The solutions to the Eq. (10) can be written in the form

$$\mu_{1,2} = A\left[2\cos^2\frac{\phi}{2} - B \pm 2\sqrt{\cos^2\frac{\phi}{2}\left(\cos^2\frac{\phi}{2} - B\right)}\right], \tag{11}$$

where

$$A = \frac{(z-1)(z+Q-1)}{z^2} \quad \text{and} \quad B = \frac{z(z+Q-2)}{(z-1)(z+Q-1)}. \tag{12}$$

Let us now study the Yang-Lee zeros of the ferromagnetic Potts model, i.e., $J > 0$ and $z > 1$. For $Q > 1$ one can easily find that $B > 1$, hence, all solutions (11) are complex conju-

[1] Zeros of the partition function considered as a function of complex Q.

[2] The results will be published elsewhere.

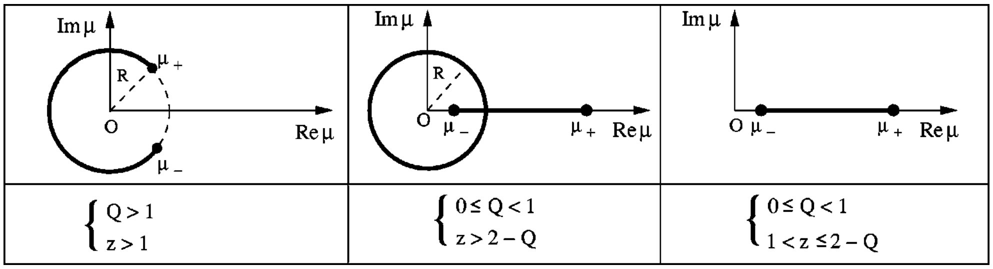

FIG. 2. A schematic representation of the Yang-Lee zeros of the 1D ferromagnetic Potts model. Here $R=(z+Q-2)/z$ and $\mu_{\pm}$ are defined in Eq. (14).

gate and lie on an arc of a circle with radius $R=(z+Q-2)/z$. If $1<Q<2$ then $R<1$ (the arc lies inside the unit circle), if $Q>2$ then $R>1$ (the arc lies outside the unit circle [9]) and $R=1$ [5] for $Q=2$ (Ising model). Writing μ in the exponential form $\mu=Re^{i\theta}$, one can find

$$\cos\frac{\theta}{2}=\sqrt{\frac{(z-1)(z+Q-1)}{z(z+Q-2)}}\cos\frac{\phi}{2}. \tag{13}$$

From Eq. (13) one can see that there are no solutions with arguments in the interval $0<\theta<\theta_0$, where $\theta_0=2\arccos\sqrt{B^{-1}}$. This is the well known gap [21] in the distribution of Yang-Lee zeros. It points to the absence of phase transitions in a 1D ferromagnetic Potts model for $Q>1$ at any real temperature. This is in good agreement with recent studies by Glumac and Uzelac [14], and Kim and Creswick [9], where the Yang-Lee zeros of the 1D Potts model was studied by using the transfer matrix method. Comparing our formula (13) with the corresponding formula (14) of Ref. [9], one can see that the argument ϕ of the derivative in our method is nothing but the difference in the arguments of two maximal eigenvalues in the transfer matrix method. It follows from Eqs. (11) and (13) that the Yang-Lee edge fields correspond to $\phi=0$ and have the form

$$\mu_{\pm}=\frac{1}{z^2}\{\sqrt{(z-1)(z+Q-1)}\pm\sqrt{1-Q}\}^2, \tag{14}$$

and $\mu_{\pm}$ are complex for $Q>1$.

$B<1$ when $Q<1$ and it is positive or negative depending on z. Hence, B is negative when $z\leqslant 2-Q(Q<1)$, and all values of μ's in Eq. (11) are real and lie between μ_- and μ_+, where $\mu_{\pm}>0$.

$0<B<1$ when $z>2-Q(Q<1)$, and all values of μ's in Eq. (11) are either real or complex depending on ϕ. For $0<\phi<\phi_0$, where $\phi_0=2\arccos\sqrt{B}$, solutions (11) are real and lie in the interval $[\mu_-,\mu_+]$. For $\phi_0<\phi<\pi$ the solutions (11) are complex conjugate and lie on the circle with radius $R=(z+Q-2)/z$.

For $Q>1$, the differentiation of both sides of Eq. (13) with respect to μ and ϕ gives the density of Yang-Lee zeros $g(\theta)$ in the form

$$g(\theta)=\frac{1}{2\pi}\frac{\left|\sin\frac{\theta}{2}\right|}{\sqrt{\sin^2\frac{\theta}{2}-\sin^2\frac{\theta_0}{2}}}. \tag{15}$$

From Eq. (15) it follows that the density function $g(\theta)$ for $Q>1$ diverges in the gap points $\mu_{\pm}$ with the exponent $\sigma=-\frac{1}{2}$, i.e., $g(\theta)\propto|\theta-\theta_0|^{\sigma}$ when $\phi\to 0$ or $\theta\to\theta_0$.

For $Q<1$, the corresponding density function $g(\mu)$ may be obtained by differentiation of both sides of Eq. (10),

$$g(\mu)=\frac{1}{2\pi\mu}\frac{|\mu-\sqrt{\mu_+\mu_-}|}{\sqrt{(\mu_+-\mu)(\mu-\mu_-)}}. \tag{16}$$

One can see that $g(\mu)$ diverges in the points $\mu_{\pm}$, i.e., $g(\mu)\propto|\mu-\mu_{\pm}|^{\sigma}$, with the exponent $\sigma=-\frac{1}{2}$. Thus, for $Q<1$ the density function of the Yang-Lee zeros of the 1D Q-state Potts ferromagnetic model has singularities only at points $\mu_{\pm}$ [Eq. (14)] with the edge singularity exponent $\sigma=-\frac{1}{2}$. The same is true for the antiferromagnetic case.

Glumac and Uzelac [14] considered for $Q<1$ case the contribution of the third eigenvalue of the transfer matrix (λ_2 in their notations). We want to note that for the 1D Potts model the transfer matrix method gives three and more eigenvalues only for $Q>2$; hence, the third and other eigenvalues should be neglected for $Q<2$. In this case the study of two maximal eigenvalues gives the same results as does our method. The summary of the results of the 1D ferromagnetic Potts model is given in Fig. 2. It is interesting to note that the argument ϕ of the derivative in Eq. (9) corresponds to the argument of the maximal eigenvalue in the transfer matrix method.

The antiferromagnetic case may be studied in the same manner. The results are shown in Fig. 3.

IV. YANG-LEE ZEROS OF THE *Q*-STATE POTTS MODEL ON A BETHE LATTICE ($Q>1$)

Let us, at first, consider the ferromagnetic Potts model on the Bethe lattice with coordination number $\gamma=3$. In this case the system (8) may be studied analytically for neutral fixed points ($k=1$). The exclusion of x from both parts of Eqs. (8) gives the following equation:

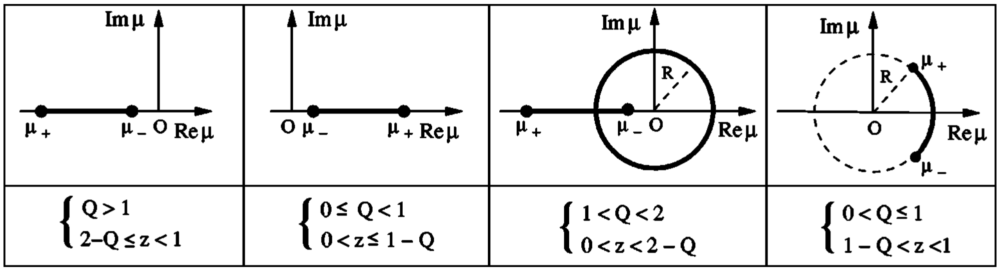

FIG. 3. A schematic representation of the Yang-Lee zeros of the 1D antiferromagnetic Potts model. Here $R=(2-Q-z)/z$ and $\mu_\pm$ are defined in Eq. (14).

$$z^3(Q-1)\mu^2-\left[2(Q-1)y(4\cos\phi+1)+y^2\left(4e^{i\phi}\sin^2\frac{\phi}{2}+1\right)-2(Q-1)^2\right]\mu+(z+Q-2)^3=0, \tag{17}$$

where $y=(z-1)(z+Q-1)/2$ and $\phi\in[0,2\pi]$. This equation describes the border of the metastability region in the complex μ plane (Fig. 4). The dashed areas in Fig. 4 show the metastability regions for different temperatures. Inside the metastability region there are two attracting fixed points and there is only one outside of it. The other fixed points are repelling. Note that such a behavior is valid for any γ. At $\mu_\pm$ at least two of the fixed points become neutral. It will be shown below that at $\mu_\pm$ the magnetization function (7) is singular and $\mu_\pm$ correspond to the Yang-Lee edge singularity points. These are solutions to Eq. (17) for $\phi=0$ and the critical temperature may be obtained from the condition $\mu_+=\mu_-$. It follows from Eq. (17) that the edge singularity points lie on a circle with radius $R_\mu^2=(z+Q-2)^3/z^3(Q-1)$. Since there is no phase transition on the boundary of a metastability region it will not give rise to zeros of the partition function [19]. The metastability region in the complex μ plane points to the existence of the first-order phase transition for complex magnetic fields. The problem of finding the Yang-Lee zeros of models with first-order phase transitions attracted much attention for many years. Recently, Biskup *et al.* showed that the position of partition function zeros is related to the phase coexistence lines in the complex planes [22]. Dolan *et al.* used this approach to study Fisher zeros for the Ising and Potts models on nonplanar (thin) regular random graphs. It is interesting to note that the locus of Fisher zeros on a Bethe lattice is identical to the corresponding random graph [23]. For our models the phase coexistence line is defined as a line in the complex plane, where the absolute values of the recursive function derivatives in two attracting fixed points are equal (see also Ref. [16]). Our numerical study showed that the phase coexistence line for $T>T_c$ is an arc of a circle with radius R_μ ending at the edge singularity points (the dashed line in Fig. 4).

Let us now study the analytical properties of the magnetization function (7) to prove that the edge singularity points $\mu_\pm$ correspond to its singularities. The fixed point equation of the Potts-Bethe mapping (5) may be written in the form

$$\mu=x^{\gamma-1}\frac{(Q-1)x-(z+Q-2)}{1-zx}. \tag{18}$$

For continuity at $\mu\neq 0$, x is defined to be equal to $(Q-1)/(z+Q-2)$ at $\mu=0$. Considering x as a function of μ, one can study the singularities of $x(\mu)$ that also correspond to singularities of $\bar{M}(\mu)$. The singular points of $x(\mu)$ are $\mu=0$, $\mu=\infty$, and $\mu(x_\pm)$, $x_\pm$ being the points where the derivative $\partial x/\partial\mu$ is infinite. $x_\pm$ satisfies the equation

$$x^2+\frac{z(z+Q-2)(2-\gamma)-\gamma(Q-1)}{z(Q-1)(\gamma-1)}x+\frac{z+Q-2}{z(Q-1)}=0. \tag{19}$$

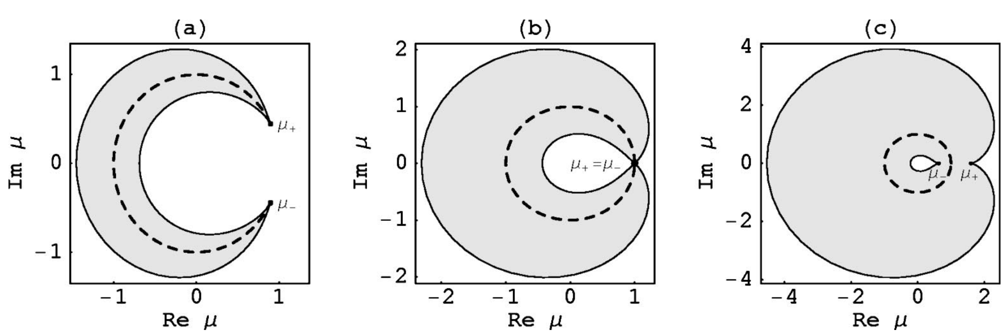

FIG. 4. Metastability regions and Yang-Lee zeros of the Q-state Potts model on the Bethe lattice with coordination number $\gamma=3$ and $Q=2$. The solid lines correspond to the boundary of metastability regions (gray filled areas). Dashed lines present an arc or circles of radius $R_\mu^2=(z+Q-2)^\gamma/z^\gamma(Q-1)^{\gamma-2}$ and correspond to Yang-Lee zeros. (a) $T>T_c(z=1.8)$, $\mu_\pm$ are Yang-Lee edge singularity points; (b) $T=T_c$ critical point $(z=z_c=3)$; (c) $T<T_c(z=6)$, $\mu_\pm$ are spinodal points of the model. For more details see the text.

Equation (19) is obtained after differentiating both sides of Eq. (18) with respect to μ and x from the condition that $\partial\mu/\partial x$ vanishes. Note that Eq. (19) may be derived from the system (8) for $k=1$ and $\phi=0$ by excluding μ from both the equations. At the singularity points of $\bar{M}(\mu)$ the recursive function has two neutral fixed points with $\phi=0$ and the others are repelling.

The critical temperature may be obtained from the condition $\mu_+=\mu_-$ or by setting the discriminant of the quadratic equation (19) to zero,

$$z_c=\begin{cases}\frac{1}{2}[2-Q+\sqrt{(Q-2)^2+4(Q-1)\gamma^2/(\gamma-2)^2}] & \text{for} \quad Q>1,\\ 1-Q & \text{for} \quad Q<1.\end{cases} \tag{20}$$

For $Q>1$ and $1<z<z_c$ the solutions $x_\pm$ are complex conjugate numbers with modulus $R_x=\sqrt{(z+Q-2)/z(Q-1)}$ and for $z>z_c$ they become real valued. For $Q<1$ the solutions $x_\pm$ are always real numbers. Substituting $x_\pm=R_xe^{i\alpha_\pm}$ into Eq. (18) one finds, after some algebra

$$\mu_\pm=R_\mu e^{i[\tilde{\theta}_\pm+\alpha_\pm(\gamma-1)]}, \tag{21}$$

where

$$R_\mu=R_x^\gamma(Q-1), \quad \tan\frac{\tilde{\theta}_\pm}{2}=\frac{1+zR_x}{1-zR_x}\tan\frac{\alpha_\pm}{2},$$

and

$$\cos\alpha_\pm=\frac{(\gamma-2)z^2R_x^2+\gamma}{2(\gamma-1)zR_x}. \tag{22}$$

Note that the fixed point equation $f(x)=x$ and the magnetization $\bar{M}(\mu)$ are invariant under the transformation $G:\{\mu\to R_\mu^2/\mu, \ x\to R_x^2/x\}$. Moreover, for the magnetization function one has

$$\bar{M}(\mu)=-\bar{M}\left(\frac{R_\mu^2}{\mu}\right). \tag{23}$$

The Yang-Lee zeros of the Potts model on the Bethe lattice may be obtained also analytically by studying the analytical properties of the magnetization function (7). This was done for the first time by Bessis, Drouffe, and Moussa [24] for the Ising model on the Bethe lattice. Making use of singularities of $x(\mu)$ in analogy with Ref. [24], a careful analysis of the analytic properties of the function $\bar{M}_\mu$ Eq. (7) for $Q>1$ gives the following picture for the Yang-Lee zeros.

(a) *For* $z<z_c$: $\bar{M}(\mu)$ is analytic in the complex plane cut along an arc of the circle with radius $R_\mu^2=(z+Q-2)^\gamma/z^\gamma(Q-1)^{\gamma-2}$, which contains the point $-R_\mu$ and is limited by the points $\mu_\pm$, which are complex conjugates. Due to relation (23), the discontinuity along the cut is real and never vanishes except at the Yang-Lee edge singularity points $\mu_\pm$. This is in a good agreement with the Yang-Lee theory [5]. The density of zeros $g(R_\mu,\theta)$ may be calculated from

$$\lim_{r\to R_\mu+}\bar{M}(\mu)|_{\mu=re^{i\theta}}-\lim_{r\to R_\mu-}\bar{M}(\mu)|_{\mu=re^{i\theta}}=-4\pi g(R_\mu,\theta). \tag{24}$$

Our numerical study of Eq. (24) shows that, close to the edge singularity points $\mu_\pm=e^{\pm i\theta_0}$, $g(R_\mu,\theta)$ has an exponential behavior $g(R_\mu,\theta)\propto|\theta-\theta_0|^\sigma$ and $\theta_0(T)\propto(T-T_c)^\Delta$, where $\sigma=1/2$ for $z<z_c$, $\sigma=1/3$ ($\sigma=1/\delta$) for $z=z_c$ and $\Delta=\frac{3}{2}$ ($\Delta=\beta\delta$). Our results are in a good agreement with those of Refs. [25,21], where it was shown that the σ exponent is universal and is always equal to 1/2 for $T>T_c$ and $\sigma=1/\delta$ for $T=T_c$ in ferromagnetic models on lattices with spatial dimension $d>6$.

(b) *For* $z>z_c$ *the function* $\bar{M}(\mu)$ *is split into two different functions*. $\bar{M}_+(\mu)$ defined for $|\mu|<R_\mu$ and $\bar{M}_-(\mu)$ defined for $|\mu|>R_\mu$. The function $\bar{M}_+(\mu)$ can be analytically continued outside the circle $|\mu|=R_\mu$ into the z plane cut along the real axes from $\mu(x_+)$ to ∞, where x_+ is the largest root of Eq. (19). The point $\mu(x_+)$ increases from unity to infinity when z increases from z_c to infinity. The discontinuity of $\bar{M}_+(\mu)$ across the cut is purely imaginary and does not change the sign. Hence, it has no influence on the physical properties of the model and the Yang-Lee zeros lie on the circle $|\mu|=R_\mu$. The points $\mu_\pm$ correspond to the boundary of metastability for real magnetic fields and the plots of the iteration function $f(x)$ at these points are given in Fig. 5.

V. COMPLEX MAGNETIC FIELD METASTABILITY REGIONS

In the preceding section the neutral fixed points of the recursive function $f(x)$, Eq. (5) was considered only. It was found that the set of values of the magnetic field, for which the recursive function (5) has at least one neutral fixed point, gives the boundary of the metastability region in the complex μ plane. Inside it, there are two attracting fixed points and others are repelling. Numerical experiments show that in the μ plane the recursive function Eq. (5) has a complex behavior with period doubling bifurcations. The question arises: What will happen to the metastability region if neutral periodical points of period $k\geqslant1$ are also considered? To answer this question, one has to study the system (8) for any $k\geqslant1$. Since it is impossible to solve the system (8) directly for k

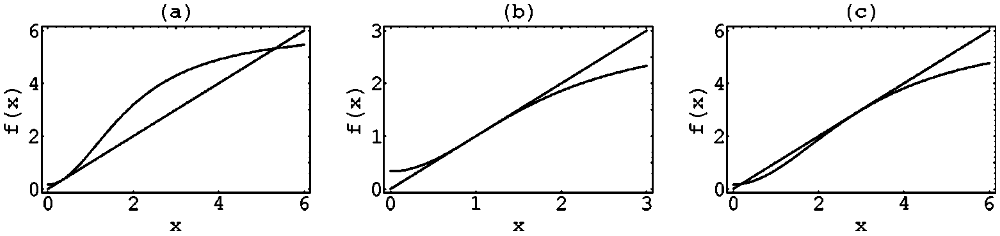

FIG. 5. Plots of $f(x)$ function (5) indicating the existence of neutral fixed points for different temperatures and magnetic fields at $Q=2$ and $\gamma=3$. (a) $T<T_c(z=6)$ and $\exp(h/kT)=\mu_-$; (b) $T=T_c$ $(z=3)h=0$; (c) $T<T_c(z=6)$, and $\exp(h/kT)=\mu_+$.

$\geqslant 1$ and $\gamma\geqslant 3$ analytically and even numerically for large k and γ, the method developed in Ref. [26] is used. It gives a numerical algorithm for searching neutral periodical points for the recursive functions such as Eq. (5) and is based on the theory of complex dynamical systems and the well known fact that the convergence of iterations to neutral periodical points is very weak and irregular, i.e., one has to make a number of iterations in order to approach a neutral periodic point. The algorithm is to find all critical points of the recursive function and investigate the convergence of all the orbits started at critical points (critical orbits). If all critical orbits converge to any attracting periodical point, one says that the recursive function has only attracting and repelling periodical points. If at least one of the critical orbits does not converge, for example, after n iterations, one says that the recursive function has a neutral periodical point. Of course, the last statement is not rigorous from the strong mathematical point of view, since a weak convergence to an attracting periodical point is also possible. Depending on the choice of n and ε (the accuracy of approaching an attracting periodical point), the resulting picture on the μ plane may change. Our simulation experiments show that $n=700$ and $\varepsilon=10^{-5}$ are optimal values and the data generated with this algorithm do not qualitatively change when n and/or ε differ from their

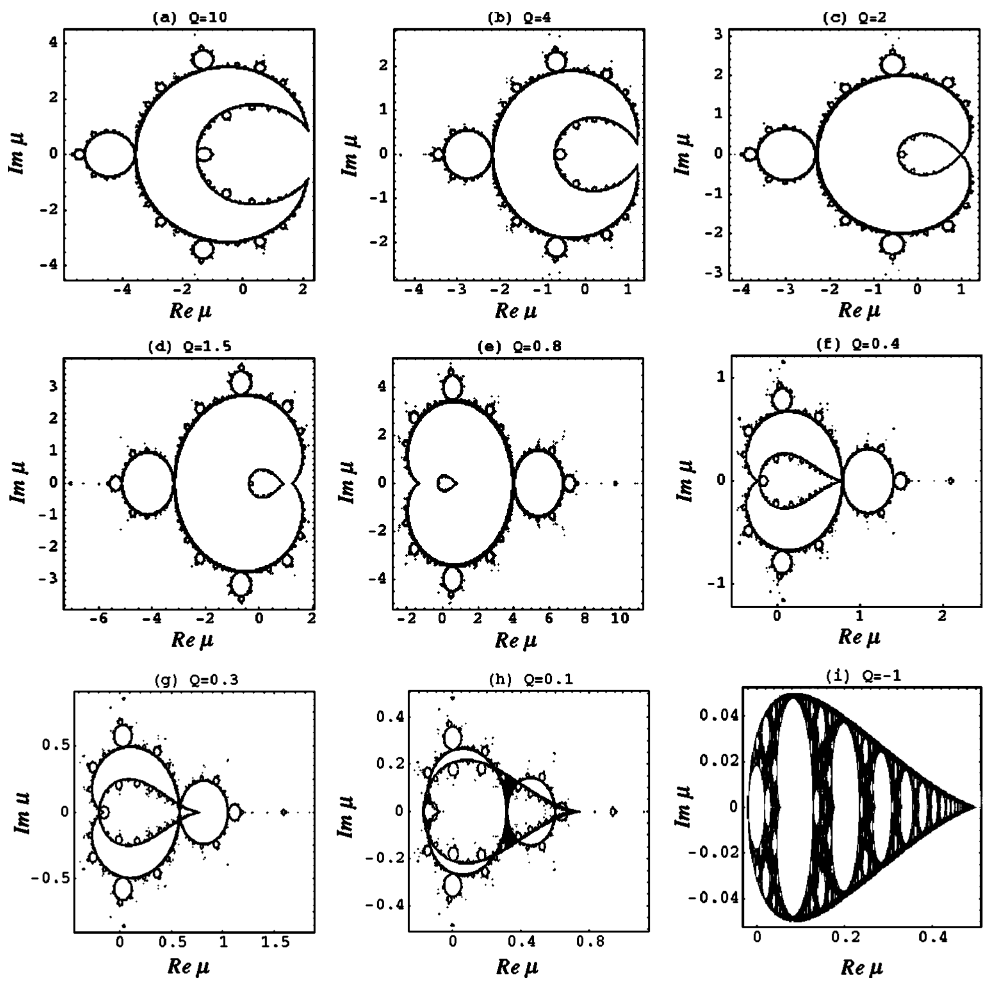

FIG. 6. The dynamics of metastability regions of the Q-state Potts model on the Bethe lattice with coordination number $\gamma=3$ and $z=3$ for different values of Q. For more details see the text.

optimal values. For more details of the method and the C++ program code see Refs. [26,27].

One can easily find all critical points of the mapping f from Eq. (5). The critical points are $x=0$ with multiplicity $\gamma-2$ and $x=\infty$ with multiplicity $\gamma-2$. The degree of our mapping f is $d=(\gamma-1)$. It may be shown that these are the only critical points of the mapping f [26]. Hence, one has to consider only the orbits of the points $x_0=0$ and $x_0=\infty$. For numerical calculations it is convenient to start iterations at the points $x_1=f(0)=1/z$ and $x_1=f(\infty)=(z+Q-2)/(Q-1)$.

In Fig. 6 the dynamics of the metastability regions of the Q-state Potts model on the Bethe lattice with coordination number $\gamma=3$ and $z=3$ is shown depending on Q. We have experimental evidence that in white regions all critical orbits converge. Figures 6(a)–6(d) show the metastability regions for the case $Q>1$. It is seen that the sets of black points are similar to the boundary of the Mandelbrot set of the quadratic mapping $z\rightarrow z^2+c$ [Figs. 6(c),6(d)]. This fact is known as the universality of the Mandelbrot set [28]. The metastability region of the Ising model ($Q=2$) at the critical temperature is shown in Fig. 6(c). It intersects the positive semiaxis at $\mu=1$, which is an evidence of the existence of real temperature second-order phase transition in conformity with exact calculations [18]. The $Q<1$ case is shown in Figs. 6(e)–6(i). One can see that Fig. 6(e) resembles a mirror reflection of the Mandelbrot set boundary of Figs. 6(d),6(e). It is worth noting that for $Q=1$ and $\gamma=3$ the Bethe-Potts mapping becomes a quadratic one [29], and our numerical method fails [26]. By lowering Q the metastability regions become more and more complicated, Figs. 6(f)–6(i). Note that the dynamics of the metastability regions remains the same if one fixes Q and changes the temperature.

VI. CONCLUSIONS

We showed that the stability analysis of attracting fixed points of the recurrence relation, i.e., the condition that the absolute values of derivatives of the governing recurrence relation in two attracting fixed points are equal at a phase transition, may be successfully applied to a study of zeros of the partition function. Note that in the one-dimensional case the condition of a phase transition is equivalent to the condition of the existence of neutral fixed points. It will be interesting to check Monroe's conjecture that at a phase transition the box counting dimension of the Julia set of the governing recurrence relation is a minimum for zeros of the partition function [13]. We suppose that the box counting dimension of the Julia set of the recurrence relation should be a minimum for zeros of the partition function also. This may serve as one more criterion for studying zeros of the partition function for models on recursive lattices.

In conclusion, we observe that numerical methods proposed in this paper are generic and may be used for the investigation of zeros of the partition function and metastability regions of other models on recursive lattices.

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